

## INTERLAMINAR STRESSES OF LAMINATED COMPOSITE JOINTS WITH DOUBLE COVER PLATES†

FO-VAN CHANG

Department of Engineering Mechanics, Tsinghua University, Peking, China

(Received 28 October 1988)

**Abstract**—Interlaminar stresses for joints with double cover plates are found by superposing two parts based upon the energy method. As the series encountered can be summed, our solutions are in closed form. The final result is checked exactly by the load the joint has to carry.

### 1. INTRODUCTION

Let two identical bars of orthotropic material with rectangular cross-section  $bh_1$  be joined lengthwise. They are glued together by two identical metallic cover plates of cross-section  $bh$ , forming a laminated composite joint with double cover plates of length  $2l$  (Fig. 1). The tensile force  $P$  is transmitted to the cover plates by the interlaminar stresses in the adhesive surfaces. These are concentrated near the ends of cover plates and often decide the strength of the joint by causing delamination.  $E$ ,  $G$ ,  $\mu$  are the elastic constants of the metal cover plates. For orthotropic bars, one principal direction is along the length with elastic constant  $E_1$ , the others being  $E_2$ ,  $G_1$  and  $\mu_{12}$ .

The interlaminar stresses of this joint can be solved by superposing that of the following two parts.

(1) The two bars in Fig. 1 are continuous, forming a laminated composite bar (Fig. 2a) and its interlaminar stresses are the first part. Transforming its cross-section into one of the same material as the cover plates, for  $E_1 < E$  we get an *I*-shaped cross-section (Fig. 2b) with the area equal to  $2bh + b(E_1/E)h_1$ . At distant cross-sections from the ends of the cover plate, for instance at the middle cross-section, the stresses in the cover plates and bar are, respectively (Fig. 2a)

$$\sigma_x = \frac{P}{2bh + \frac{E_1}{E}bh_1}, \quad \sigma'_x = \frac{P}{2bh + \frac{E_1}{E}bh_1} \cdot \frac{E_1}{E}. \quad (a)$$

(2) To remove the uniform tensile stress  $\sigma'_x$  in the middle cross-section of the orthotropic bar (Fig. 2a), apply a pair of uniform compressive stresses  $\sigma'_x$  at the two opposite surfaces of the gap in the joint (Fig. 1). Figure 3 shows the left half of the joint, the tensile stress at the middle cross-section of cover plate being  $\sigma'_x(h_1/2h)$  to keep it in equilibrium. Its interlaminar stresses form the second part. The corresponding axial compressive and tensile forces are:

$$-\frac{Ph_1}{2h + \frac{E_1}{E}h_1} \cdot \frac{E_1}{E}, \quad \frac{Ph_1}{2\left(2h + \frac{E_1}{E}h_1\right)} \cdot \frac{E_1}{E}. \quad (b)$$

### 2. THE INTERLAMINAR STRESSES OF PART ONE

Figure 4 shows the interlaminar stresses  $\tau_0$  and  $\sigma_0$ , which are expressed by a sine and cosine series with  $a_n$  and  $b_n$  to be determined:

† Project supported by the National Natural Science Foundation of China.

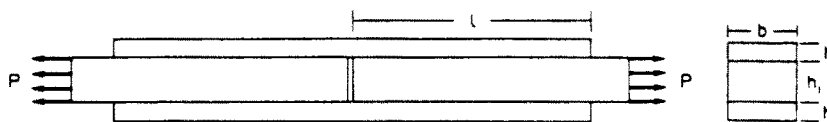


Fig. 1.

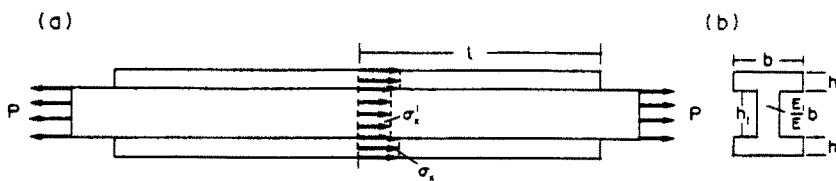


Fig. 2.

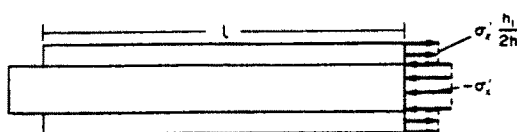


Fig. 3.

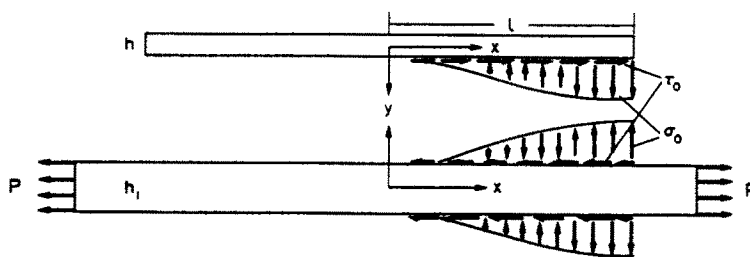


Fig. 4.

$$\tau_0 = \sum_{n=1} a_n \sin \frac{n\pi x}{l}, \quad \sigma_0 = \sum_{n=1} b_n \cos \frac{n\pi x}{l}. \tag{1}$$

As  $\sigma_0$  will form a couple,  $b_0 = 0$ .

From the cover plate take an element  $dx$  (Fig. 5) upon which are acting the axial force  $S$  and shearing force  $Q$ . Its equilibrium gives

$$\frac{dS}{dx} = -b\tau_0, \quad \frac{dQ}{dx} = -b\sigma_0, \quad Q = \frac{\tau_0}{2}bh. \tag{2}$$

From (2) it follows that

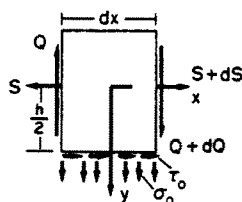


Fig. 5.

$$\sigma_0 = -\frac{h}{2} \cdot \frac{d\tau_0}{dx}. \quad (3)$$

Thus along the adhesive surface  $\sigma_0$  is proportional to the slope of the  $\tau_0$  curve negatively at the very point. From (1) and (3) we have the relation between  $a_n$  and  $b_n$ :

$$b_n = -\frac{h\pi}{2l} a_n n. \quad (4)$$

It enables us to express the stress components of the cover plates and bar by  $a_n$ . Integrating the first equation of (2) we get

$$S = -\frac{bl}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right).$$

Now the normal stress component  $\sigma_x$  of the cover plate is

$$\sigma_x = \frac{S}{bh} = -\frac{l}{\pi h} \sum_{n=1}^{\infty} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right). \quad (5a)$$

Substituting  $\sigma_x$  in the integral from the equilibrium equation

$$\int_y^{h/2} \frac{\partial \sigma_x}{\partial x} dy + \int_{\tau_{xy}}^{\tau_0} \frac{\partial \tau_{xy}}{\partial y} dy = 0,$$

we get:

$$\tau_{xy} = \left( \frac{1}{2} + \frac{y}{h} \right) \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}. \quad (5b)$$

And from another equilibrium equation we get

$$\sigma_y = -\left( \frac{1}{4} + \frac{y}{h} + \frac{y^2}{h^2} \right) \frac{h\pi}{2l} \sum_{n=1}^{\infty} a_n n \cos \frac{n\pi x}{l}. \quad (5c)$$

In a similar manner the stress components of the bar are obtained as

$$\begin{aligned} \sigma'_x &= \frac{P}{bh_1} + \frac{2l}{\pi h_1} \sum_{n=1}^{\infty} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right), \\ \tau'_{xy} &= \frac{-2y}{h_1} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}, \\ \sigma'_y &= -\frac{\pi h}{2l} \left( 1 + \frac{h_1}{2h} - \frac{2y^2}{hh_1} \right) \sum_{n=1}^{\infty} a_n n \cos \frac{n\pi x}{l}. \end{aligned} \quad (6)$$

Thus all the stress components are expressed by  $a_n$ , which will be determined by the Principle of Least Work. Let  $U$  be the total strain energy of this composite bar and the coefficients  $a_n$  should be such as to minimize  $U$ , namely

$$\begin{aligned} \frac{\partial U}{\partial a_n} = & 2 \int_0^l \int_{-h/2}^{h/2} \left\{ \frac{1}{E} \left[ \sigma_x \frac{\partial \sigma_x}{\partial a_n} + \sigma_y \frac{\partial \sigma_y}{\partial a_n} - \mu \left( \sigma_x \frac{\partial \sigma_y}{\partial a_n} + \sigma_y \frac{\partial \sigma_x}{\partial a_n} \right) \right] \right. \\ & \left. + \frac{1}{G} \tau_{xy} \frac{\partial \tau_{xy}}{\partial a_n} \right\} b \, dx \, dy + \int_0^l \int_{-h_1/2}^{h_1/2} \left\{ \frac{\sigma'_x}{E_1} \frac{\partial \sigma'_x}{\partial a_n} + \frac{\sigma'_y}{E_2} \frac{\partial \sigma'_y}{\partial a_n} - \frac{\mu_{12}}{E_1} \left[ \sigma'_x \frac{\partial \sigma'_y}{\partial a_n} + \sigma'_y \frac{\partial \sigma'_x}{\partial a_n} \right] \right. \\ & \left. + \frac{\tau'_{xy}}{G_1} \frac{\partial \tau'_{xy}}{\partial a_n} \right\} b \, dx \, dy = 0. \quad (7) \end{aligned}$$

Substituting the six stress components in (7) we get, after lengthy calculation,

$$a_n = -C \frac{n \cos n\pi}{n^4 + 2\eta n^2 + p^2}, \quad (8)$$

in which

$$\begin{aligned} p^2 &= \frac{\frac{1}{\pi^2} \left( \frac{1}{E} + \frac{1}{E_1} \frac{2h}{h_1} \right)}{\frac{\pi^2 h^4}{4 l^4} \left[ \frac{1}{5E} + \frac{1}{E_2} \frac{h_1}{h} \left( \frac{1}{2} + \frac{1}{3} \frac{h_1}{h} + \frac{1}{15} \frac{h_1^2}{h^2} \right) \right]}, \\ 2\eta &= \frac{\frac{1}{3} \frac{\mu}{E} - \frac{\mu_{12}}{E_1} \left( 1 + \frac{h_1}{3h} \right) + \frac{1}{3} \left( \frac{1}{G} + \frac{1}{2G_1} \frac{h_1}{h} \right)}{\frac{\pi^2 h^2}{4 l^2} \left[ \frac{1}{5E} + \frac{1}{E_2} \frac{h_1}{h} \left( \frac{1}{2} + \frac{1}{3} \frac{h_1}{h} + \frac{1}{15} \frac{h_1^2}{h^2} \right) \right]}, \\ C &= \frac{\frac{2}{\pi} \left\{ \frac{1}{E_1} \frac{h}{l} \sigma^* + \frac{1}{\pi} \left( \frac{1}{E} + \frac{2h}{E_1 h_1} \right) \sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi \right\}}{\frac{\pi^2 h^4}{4 l^4} \left[ \frac{1}{5E} + \frac{1}{E_2} \frac{h_1}{h} \left( \frac{1}{2} + \frac{1}{3} \frac{h_1}{h} + \frac{1}{15} \frac{h_1^2}{h^2} \right) \right]}, \quad (9) \end{aligned}$$

in which  $\sigma^* = P/bh_1$ . As can be seen from (9), when the cover plates and bar are of the same isotropic material,  $p$ ,  $2\eta$  and  $C$  will be free from elastic constants. As a result,  $\tau_0$  and  $\sigma_0$  do not depend upon the elastic constants. When the cover plates are of the same orthotropic material as the bar, as often happens in wooden structures, then in the denominators of (9)  $E$  has to be changed into  $E_2$  and in the numerators  $E$ ,  $G$ ,  $\mu$  have to be changed into  $E_1$ ,  $G_1$ ,  $\mu_{12}$ .

From (8) and (1) the interlaminar shearing stresses  $\tau_0$  become:

$$\tau_0 = -C \sum_{n=1}^{\infty} \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^4 + 2\eta n^2 + p^2}. \quad (10)$$

As series (10) can be summed,  $\tau_0$  and  $\sigma_0$  are both in closed form. For  $p > \eta$  and for  $\beta\pi$  to be a comparatively large value, as usually occurs  $\tau_0$  becomes:

$$\tau_0 = \frac{C\pi}{4\beta\gamma \sinh \beta\pi} \left\{ \sin \gamma\pi \sinh \frac{\beta\pi}{l} x \cos \frac{\gamma\pi}{l} x - \cos \gamma\pi \cosh \frac{\beta\pi}{l} x \sin \frac{\gamma\pi}{l} x \right\}, \quad (11)$$

in which  $\beta = \sqrt{\frac{1}{2}(p+\eta)}$ ,  $\gamma = \sqrt{\frac{1}{2}(p-\eta)}$ . From (3),  $\sigma_0$  is given by

$$\sigma_0 = -\frac{h}{2} \frac{d\tau_0}{dx} = \frac{C\pi^2}{4\beta\gamma \sinh \beta\pi} \cdot \frac{h}{2l} \left\{ (\gamma \sin \gamma\pi + \beta \cos \gamma\pi) \sinh \frac{\beta\pi}{l} x \sin \frac{\gamma\pi}{l} x - (\beta \sin \gamma\pi - \gamma \cos \gamma\pi) \cosh \frac{\beta\pi}{l} x \cos \frac{\gamma\pi}{l} x \right\}. \quad (12)$$

Near the ends of the cover plate when  $\sinh(\beta\pi/l)x = \cosh(\beta\pi/l)x$ , a further simplification can be made as follows:

$$\tau_0 = \frac{C\pi}{4\beta\gamma} \cdot \frac{\sinh \frac{\beta\pi}{l} x}{\sinh \beta\pi} \sin \gamma\pi \left(1 - \frac{x}{l}\right). \quad (13)$$

$$\sigma_0 = \frac{C\pi^2}{4\beta\gamma} \cdot \frac{h}{2l} \frac{\sinh \frac{\beta\pi}{l} x}{\sinh \beta\pi} \left\{ \gamma \cos \gamma\pi \left(1 - \frac{x}{l}\right) - \beta \sin \gamma\pi \left(1 - \frac{x}{l}\right) \right\}. \quad (14)$$

The shearing force transmitted by the cover plate is

$$b \int_0^l \tau_0 dx = \frac{Cbh_1}{4p\beta} \frac{h_1}{l}. \quad (15)$$

The value  $C$  in the above equations has to be determined from the third equation of (9) which includes the series  $\sum_{n=1} (a_n/n) \cos n\pi$ . Substituting (8) in it, we have for  $p > \eta$  with  $\beta\pi$  pretty large:

$$\sum_{n=1} \frac{a_n}{n} \cos n\pi = -C \sum_{n=1} \frac{1}{n^4 + 2\eta n^2 + p^2} = -C \left( \frac{\pi}{4p\beta} - \frac{1}{2p^2} \right). \quad (16)$$

### 3. THE INTERLAMINAR STRESSES OF PART TWO

Differing from Fig. 4, Fig. 6 corresponds to the left half of the joint with the origin taken at the left end of the cover plate. We still use (1) to denote the interlaminar stresses  $\tau_0$  and  $\sigma_0$ .

The equations obtained from the equilibrium of an element  $dx$  of the cover plate are the same as (2). This results in two equations, the same as (3) and (4). The cover plate has its axial force equal to

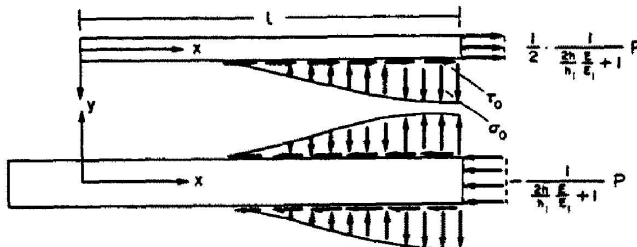


Fig. 6.

$$S = -\frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right) + \frac{1}{2} \frac{P}{\frac{2h}{h_1} \frac{E}{E_1} + 1}.$$

The normal stress  $\sigma_x$  is

$$\sigma_x = \frac{S}{bh} = -\frac{l}{\pi h} \sum_{n=1} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right) + \frac{\sigma^*}{2} \frac{h_1}{h} \cdot \frac{1}{\frac{2h}{h_1} \frac{E}{E_1} + 1}. \quad (c)$$

It differs from (5a) of the first part only in the additional constant term and hence  $\tau_{xy}$  and  $\sigma_y$  are the same as (5b, c). Similarly, for the bar we have

$$\sigma'_x = \frac{S'}{bh_1} = \frac{2l}{\pi h_1} \sum_{n=1} \frac{a_n}{n} \left( \cos n\pi - \cos \frac{n\pi x}{l} \right) - \frac{\sigma^*}{\frac{2h}{h_1} \frac{E}{E_1} + 1}. \quad (d)$$

It also differs from  $\sigma'_x$  of the first part in the constant term, so that  $\tau'_{xy}$  and  $\sigma'_y$  remain the same as in (6). Substituting the six stress components in (7) we again have a result expressed by equation (8). Besides, of the six stress components only  $\sigma_x$  and  $\sigma'_x$  differ from those of the first part by their constant terms;  $p$  and  $2\eta$  remain the same as in equation (9) of the first part. Only the third one has to be changed, into

$$C = \frac{\frac{2}{\pi} \left\{ \frac{\sigma^*}{E_1} \cdot \frac{h}{l} \left( -\frac{h_1 E_1}{2hE} \right) + \frac{1}{\pi} \left( \frac{1}{E} + \frac{2h}{E_1 h_1} \right) \sum_{n=1} \frac{a_n}{n} \cos n\pi \right\}}{\frac{\pi^2 h^4}{4 l^4} \left[ \frac{1}{5E} + \frac{1}{E_2} \frac{h_1}{h} \left( \frac{1}{2} + \frac{h_1}{3h} + \frac{1}{15} \frac{h_1^2}{h^2} \right) \right]}. \quad (17)$$

Equation (17) differs from the third part of (9) in that  $(\sigma^*/E_1) \cdot (h/l)$  has to be multiplied by a factor

$$-\frac{h_1 E_1}{2hE}. \quad (18)$$

Since the two parts of superposition have the same values of  $p$  and  $2\eta$ , their  $\beta$  and  $\gamma$  are also same. Thus from eqns (11) and (12), it can be seen that the two different values of  $C$  constitute the difference of two parts of the interlaminar stresses. Apparently, the sum of the series  $\sum_{n=1} (a_n/n) \cos n\pi$  within the parentheses of (16) remains the same for the two parts. As a result, from (17) and the third eqn of (9), it is seen that the value of  $C$  of the second part is equal to that of first part multiplied by the factor (18). Therefore, the interlaminar stresses of the first part multiplied by this same factor give the interlaminar stresses of second part. Negative  $\tau_0$  means the direction is opposite to that shown in Fig. 6 and negative  $\sigma_0$  means compressive stress.

For cover plates and bar made of the same orthotropic material, by setting  $E$  equal to  $E_1$  in (18) we get the factor  $-(h_1/2h)$ . Multiplying the interlaminar stresses of the first part by this factor, we get the interlaminar stresses of the second part.

However, the solution of first part of superposition is for the right half of the joint, while the second part is for the left half. Owing to the symmetry of  $\sigma_0$  and anti-symmetry of  $\tau_0$ , we can easily convert the first part to the left half and superpose with the second part to get the interlaminar stresses of the joint.

## 4. NUMERICAL ILLUSTRATIVE EXAMPLES

## 4.1. Example 1

An adhesive joint with double cover plates is made of wooden bars and hard aluminium cover plates with  $h_1 = 6h$ ,  $l/h = 32$  and  $l/(2h+h_1) = l/8h = 4$ . For pine bars:  $E_1 = 10^5$  kg cm<sup>-2</sup>,  $E_2 = 0.042 \times 10^5$  kg cm<sup>-2</sup>,  $G_1 = 0.075 \times 10^5$  kg cm<sup>-2</sup>,  $\mu_{12} = 0.238$ . For hard aluminium:  $E = 7 \times 10^5$  kg cm<sup>-2</sup>,  $G = 2.69 \times 10^5$  kg cm<sup>-2</sup>,  $\mu = 0.3$ .

*First part of superposition*

From (9) and (11),  $p = 5.4121$ ,  $\eta = 3.7816$ ;  $\beta = 2.1440$ ,  $\gamma = 0.90291$ . The series sum (16) gives  $\sum_{n=1} (a_n/n) \cos n\pi = -0.050616C$ . The third equation of (9) gives  $C = 3.0459\sigma^*$ .

By (15) the shearing force transmitted by the cover plate is

$$\int_0^l b\tau_0 dx = \frac{3.0459P}{4 \times 5.4121 \times 2.144 \times \frac{6}{2}} = 0.34999P.$$

To check our calculation let us find the tensile force in the middle cross-section of the cover plate. From (a) it is equal to

$$\sigma_x bh = \frac{P}{2 + \frac{10^5}{7 \times 10^5} \times 6} = 0.35P.$$

This indicates the correctness of our calculation. Now by (11)–(14),  $\tau_0$  and  $\sigma_0$  are found and tabulated as follows.

$x$	$l$	0.98	0.96	0.94	0.92	0.9	0.85	0.8	0.75
$\tau_0$	0	0.0710 $\sigma^*$	0.1069	0.1398	0.1623	0.1763	0.1875	0.1726	0.1491
$\sigma_0$	0.0547 $\sigma^*$	0.0417	0.0303	0.0213	0.0141	0.0083	-0.0014	-0.0062	-0.0080
$x$	0.7	0.65	0.6	0.55	0.5	0.45	0.4	0.3	0.2
$\tau_0$	0.1232	0.0979	0.0757	0.0571	0.0421	0.0305	0.0216	0.0102	0.0044
$\sigma_0$	-0.0082	-0.0075	-0.0064	-0.0052	-0.0041	-0.0032	-0.0024	-0.0013	

*Second part of superposition*

The factor (18) is

$$-\frac{6 \times 10^5}{2 \times 7 \times 10^5} = -0.42857.$$

Multiplying  $\tau_0$  and  $\sigma_0$  in the above table by this factor, we get the interlaminar stresses of the second part. The shearing force transmitted by the cover plate (Fig. 6) is equal to the corresponding shearing force of the first part multiplied by this factor, namely

$$-0.42857 \times 0.34999P = -0.14999P.$$

The negative sign means its direction is opposite to that in Fig. 6. This result is confirmed by the tensile force at the end of cover plate:

$$\frac{1}{2} \cdot \frac{1}{\frac{2}{6} \cdot \frac{10^5}{7 \times 10^5} + 1} = 0.15P.$$

Adding the shearing forces transmitted by the cover plate of two parts, we have the force transmitted by the joint:

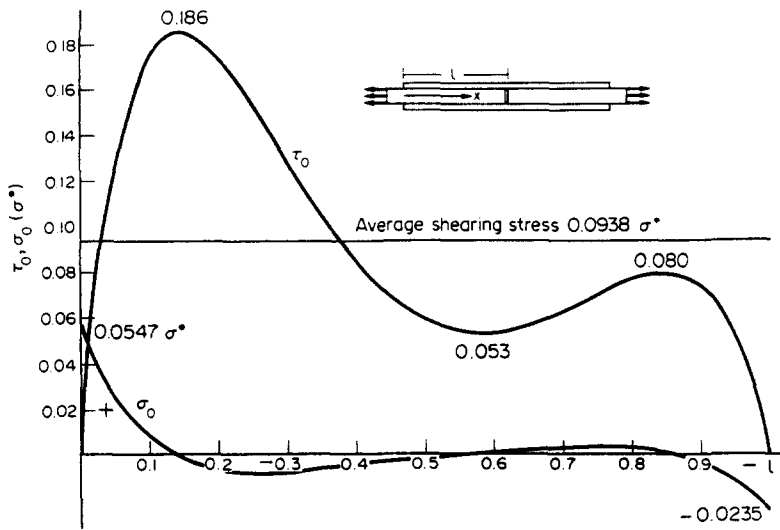


Fig. 7.

$$0.34999P + 0.14999P = 0.49998P.$$

The exact value is  $0.5P$ . Converting  $\tau_0$  and  $\sigma_0$  of the first part to the left half (Fig. 4) and superposing with that of the second part, we get the interlaminar stresses of the joint as shown in Fig. 7.

As can be seen from Fig. 7,  $\sigma_0$  is small compared with  $\tau_0$ . The reason is that for the thin hard aluminium cover plate the bending moments formed by  $\tau_0$  are rather small and require a comparatively small  $\sigma_0$  to reduce them to zero. However, the maximum shearing stress is twice as large as the average value.

4.2. Example 2

As a second example, let there be a pine adhesive joint with double cover plates,  $h/l = 1/16$ ,  $h/h_1 = 3/5$ ,  $l/(2h + h_1) = 4.36$ . The elastic constants are as given in first example.

*First part of superposition*  
 Converting equations (9) to the case for cover plates and bars made of the same orthotropic material, we get  $p = 10.471$ ,  $\eta = 7.5483$ .  $\beta, \gamma$  in (11) are respectively 3.0016 and 1.2089. The series sum  $\sum_{n=1}^{\infty} (a_n/n) \cos n\pi = -0.020428C$  and  $C = 3.5716\sigma^*$ .

To check our solution, use (15) to find the shearing force transmitted by cover plate:

$$\int_0^l b\tau_0 dx = \frac{3.5716P}{4 \times 10.471 \times 3.0016 \times \frac{5}{3} \times \frac{1}{16}} = 0.27273P.$$

The tensile force in the middle cross-section of the cover plate is, from (a),

$$\frac{P}{2 + \frac{5}{3}} = 0.27273P.$$

The identity of the two results confirms our solution. Now using (11)–(14) we get  $\tau_0$  and  $\sigma_0$  which are tabulated as follows.

$x$	$l$	0.98	0.96	0.94	0.92	0.9	0.8	0.75
$\tau_0$	0	0.0486	0.0802	0.0992	0.1088	0.1116	0.0807	0.0595
$\sigma_0$	$0.0917\sigma^*$	0.0614	0.0386	0.0215	0.0091	0.0003	-0.0137	-0.0125
$x$	0.7	0.65	0.6	0.5	0.4			
$\tau_0$	0.0415	0.0277	0.0178	0.0066	0.0020			
$\sigma_0$	-0.0099	-0.0073	-0.0051	-0.0022	-0.0008			



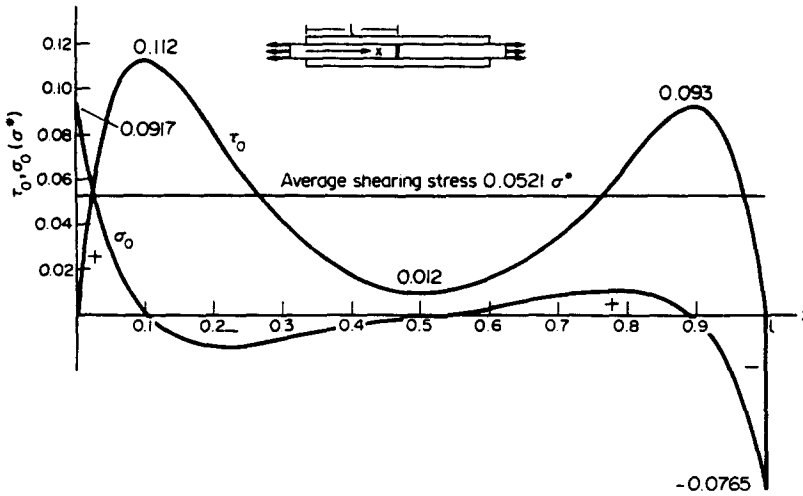


Fig. 8.

*Second part of superposition*

As factor (18) gives  $h_1/2h = -5/6$ , multiplying it by the interlaminar stresses of the above table, we get the results of the second part. The shearing force taken by the cover plate along the adhesive surface is

$$-5/6 \times 0.27273P = -0.22727P.$$

This should be equal to the tensile force at the end of the cover plate (Fig. 6) :

$$\frac{1}{2} \cdot \frac{1}{2 \times \frac{1}{3} + 1} = 0.22727P.$$

The result obtained confirms our solution. Adding the two shearing forces along the adhesive surfaces of the two parts, we get :

$$0.27273P + 0.22727P = 0.5P.$$

This is exactly equal to the load which a cover plate is required to carry. Superposing the interlaminar stresses of two parts, we have that of the joint as shown in Fig. 8.

The numerical examples show that the interlaminar normal and shearing stresses are concentrated near the ends of the cover plates. This accounts for the tearing of the cover plates from the bar starting at the ends of the cover plates, as observed from experiments with these kinds of joints.

5. DISCUSSION

The interlaminar stresses of laminated composite joints with double cover plates present a problem of practical and theoretical interest. As previous works show, it is still not satisfactorily solved. Yuceoglu and Updike (1980, 1981) regarded the adherends as beams in action, which can take into account only the bending stress  $\sigma_x$  and shearing stress  $\tau_{xy}$ . The stress component  $\sigma_y$  is, in fact, as important as the normal interlaminar stress  $\sigma_0$  we are seeking. Disregarding it in adherends leads erroneously to the interlaminar shearing stress  $\tau_0$  becoming maximum at the ends of the adhesive surface instead of zero as it should be. Another type of solution was by the finite element method and valid only for special cases of geometry and material.

Our solution is based upon the interlaminar stresses  $\tau_0$  and  $\sigma_0$  of the corresponding laminated composite bar. They are expressed by sine and cosine series as in (1). As the

composite bar consists of three slender bars, their stress components  $\sigma_x$  can be obtained by the Mechanics of Materials. The other stress components follow from equations of equilibrium. With the relation between the coefficients  $a_n$  and  $b_n$ , all the stress components can be expressed by  $a_n$ , which is determined by the Principle of Least Work. As all the series encountered can be summed, we get for this composite bar interlaminar stresses in closed form. Seeing that the two series in (1) are complete sets of functions and in the process of investigation we have not neglected anything, exact solutions have been obtained. This confirms the simple solution long taught in Mechanics of Materials and enables us to clarify how far from the ends this simple solution holds. It can also be shown that the interlaminar stresses given here require the composite bar to be sufficiently long, just as the bending theory requires the beam to be rather slender. Furthermore, when the bar is long enough, for any specific composite bar its interlaminar stresses are independent of its length. With all these considerations, we can simply go a step further by superposition to solve the joint problem.

Extensions can be made to solve the following problems:

- (1) interlaminar stresses of laminated composite lap joints and joints with a single cover plate;
- (2) thermal interlaminar stresses of laminated composite bars and joints;
- (3) the effects of an adhesive layer with a certain thickness upon interlaminar stresses of laminated composite bars and joints.

#### REFERENCES

- Erdogan, F. (1971). Analysis of elastic cover plates. *Devel. Mech.* **6**, 817–829.
- Hoff, N. J. (1944). Stresses in a reinforced monocoque cylinder under concentrated symmetric transverse loads. *J. Appl. Mech.* **11**, A-235.
- Segerlind, L. J. (1968). On the shear stress in bonded joints. *J. Appl. Mech.* **35**, 117–118.
- Timoshenko, S. (1924). Approximate solution of two dimensional problems in elasticity. *Philosophical Mag.* **47**, 1095–1104.
- Yuceoglu, U. and Updike, D. P. (1980). Stress analysis of bonded plates and joints. *J. Engng Mech. Div. ASCE* **106**, 39–56.
- Yuceoglu, U. and Updike, D. P. (1981). Bending and shear deformation effects in lap joints. *J. Engng Mech. Div. ASCE* **107**, 55–76.
- Zhang, F. (1985). Interlaminar stresses of laminated composite bars under bending, I, II. *Appl. Maths. Mech.* **7**, 8, 583–594, 699–710.
- Zhang, F. (Chang, F.-v.) (1988). Interlaminar stress of a laminated composite lap joint. *Scientia Sinica* **31**, 69–78.